Dynamics of Giant-Gravitons in the LLM geometry
and the Fractional Quantum Hall Effect

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Abstract

The LLM's 1/2 BPS solutions of IIB supergravity are known to be closely related to the integer quantum Hall droplets with filling factor $\nu = 1$, and the giant gravitons in the LLM geometry behave like the quasi-holes in those droplets. In this paper we consider how the fractional quantum Hall effect may arise in this context, by studying the dynamics of giant graviton probes in a special LLM geometry, the $AdS_5 \times S^5$ background, that corresponds to a circular droplet. The giant gravitons we study are D3-branes wrapping on a 3-sphere in S^5. Their low energy world-volume theory, truncated to the 1/2 BPS sector, is shown to be described by a Chern-Simons finite-matrix model. We demonstrate that these giant gravitons may condense at right density further into fractional quantum Hall fluid due to the repulsive interaction in the model, giving rise to the new states in IIB string theory. Some features of the novel physics of these new states are discussed.

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I. INTRODUCTION

The Quantum Hall Effect (QHE), a novel phenomenon of 2d electron gas in a strong transversal magnetic field, has been shown to have intriguing relationship with string/M-theory. In the early days, the realization of the quantum Hall systems in terms of solitonic systems in string/M theory (M-branes or D-branes plus strings etc.) is the major focus [1, 2, 3]. Recently this issue has attracted new interests in the context of AdS/CFT correspondence due to the progress in understanding of the 1/2 BPS sector on both string theory and gauge theory sides.

In a remarkable work [4], Lin, Lunin and Maldacena (LLM) found a general class of static, non-singular solutions of IIB supergravity; these solutions preserve at least 16 supercharges together with a $SO(4) \times SO(4) \times R$ global symmetry, where R is the time translation symmetry. The maximally supersymmetric solution $AdS_5 \times S^5$ is a special case of the LLM solutions. For an observer at infinity, the solutions have total energy equal to angular momentum, which is just the 1/2 BPS condition in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) in 4 dimensions. Therefore, the LLM geometries as semi-classical gravitational excitations in Type IIB string theory on $AdS_5 \times S^5$ background correspond to the 1/2 BPS sector in the dual CFT, i.e. the $\mathcal{N} = 4$ SYM in 4d.

LLM's interest in finding the gravitational dual of the 1/2 BPS sector in $\mathcal{N} = 4$ SYM was inspired by [5, 6]. In [6], Berenstein made the following observation: certain limiting procedure produces a decoupled 1/2 BPS sector in AdS/CFT correspondence and the self consistency of this decoupling procedure roots in the fact that all other degrees of freedom cost too much energy and can be integrated out. Accordingly, the study of the 1/2 BPS sector as LLM conducted provides a new testing

ground for AdS/CFT correspondence, which is more controllable on both sides than the entire theories. Also it was noted in [6] that the 1/2 BPS sector in $\mathcal{N} = 4$ SYM can be mapped to a system of 1d free fermions, which in turn can be mapped to quantum Hall droplets with filling factor $\nu = 1$ in 2d phase space. (This observation was further substantiated in refs. [8, 9, 10, 11].)

LLM geometries provide us with an ideal setting for studying the AdS/QH/SYM connection on the string theory side. This is because the LLM solutions are completely determined by the boundary value of a single function z of three non-compact spatial coordinates

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(x_1, x_2, y) with $y \geq 0$. When the boundary value of z on the two dimensional plane at $y = 0$ is taken to be $\pm 1/2$, the corresponding geometries are non-singular. This boundary value can be interpreted as the distributions for two types of point charge sources on the two dimensional boundary plane. R-R five-flux quantization infers that the areas of connected regions for $z = \pm 1/2$ on the boundary plane to be quantized in appropriate unit. Therefore, one may view the either regions with $z = 1/2$ or $z = -1/2$ on $y = 0$ plane as droplets of an incompressible fluid, like the quantum Hall fluid. The special case $AdS_5 \times S^5$ corresponds a uniform distribution of point sources in a circular disk with $z = -1/2$ on the $y = 0$ plane so it corresponds to a circular quantum Hall droplet on the plane. In this way, it is convenient to describe the $AdS_5 \times S^5$ geometry and its small deformations (the so-called bubbling AdS geometries) in terms of quantum Hall droplets [4, 7].

Since Integer Quantum Hall Effect (IQHE) ($\nu = 1$ for filling fraction) emerges naturally in the LLM geometries, one can't help but wonder whether this is merely an accidental analogy or this signals any profound physics? More precisely, one may wonder whether the knowledge of QHE in condensed matter physics can provide any insights into the dynamics of the LLM geometries. In this paper, we make progress in this direction by examining whether *Fractional Quantum Hall Effect* (FQHE) can make its way into the dynamics of the LLM geometries. In condensed matter physics, it is well-known that the quasi-particles or quasi-holes in the $\nu = 1$ quantum Hall system can condense to form new quantum Hall liquid states as an effect of interactions. For example, the simplest case of $\nu = 2/3$ FQH state can be viewed as the condensation of the quasi-holes in the $\nu = 1$ IQH fluid.

As we just mentioned, $AdS_5 \times S^5$ background is treated as a circular QH droplet in the LLM/IQH analogy. Adding a giant graviton in S^5 which is a D3-brane wrapping on a 3-sphere in S^5 [12] corresponds to a quasi-hole excitation [4] while adding a giant graviton in AdS_5 [13, 14] corresponds to a quasi-particle excitation. For simplicity and definitude, we will only examine the dynamics of quasi-hole type of giant gravitons, and treat these giant gravitons as probes to the $AdS_5 \times S^5$ background (as a special LLM geometry). Recall that the five-form flux of $AdS_5 \times S^5$ background plays the role of the constant magnetic field on the $y = 0$ plane. We will show that the low-energy effective action for the giant gravitons of quasi-hole type, when truncated by 1/2 BPS condition, exhibits all of the essential features of a QH system. In other words, the giant gravitons in the 1/2 BPS sector behave just like the charged particles in a strong magnetic field.

For one single giant graviton, the dynamics of the Landau problem for a single quasi-hole is exactly reproduces. This result is just a self-consistent check that the $AdS_5 \times S^5$ background

behaves as an unexcited $\nu = 1$ QH droplet. For many giant gravitons, if their number is much smaller than the background flux, we can ignore their back-reaction. Then the non-abelian low-energy dynamics (world-volume theory) of the giant graviton probes leads to a Chern-Simons matrix model; this matrix model is exactly the matrix description of the fluid dynamics of a system of non-relativistic charged particles moving in a constant magnetic field, i.e. the infinite-matrix model in ref. [3] for the bulk QH state, or the finite-matrix model in ref. [18] that incorporates edge excitations. Here the matrix description for the QH fluid is remarkably suitable for our purpose in the context of giant graviton dynamics: The expectation values of the diagonal elements are the positions of giant gravitons in the transverse space, while the off-diagonal elements describe open strings stretching between different giant gravitons, giving rise to the interaction between giant gravitons (as quasi-holes). Ignoring the off-diagonal elements and following the arguments in ref. [6], we will show that this matrix model in the eigenvalue basis reduces to a free fermion system, provided all of the eigenvalues are non-degenerate. It implies that the world-volume gauge symmetry of M giant gravitons is broken from $U(M)$ down to $U(1)^M$, and each giant graviton fills a state with different angular momentum j. Including off-diagonal elements will lead to *repulsive* interactions between giant gravitons, in favor of forming new incompressible fluids with fractional filling factors (FQH fluid). This is consistent with the fact that the level of the resulting Chern-Simons matrix model takes only integer values, whose inverses correspond to the fractional filling factors. In this manner, we are able to demonstrate that the 1/2 BPS dynamics of the giant gravitons wrapping on the three-sphere in S^5, which

behave like quasi-holes in an IQH droplet with $\nu = 1$, allows the formation of a ground state corresponding to a FQH liquid due to quasi-hole condensation. Moreover, when the number M of the giant gravitons is finite but large, the resulting finite-matrix model allows gapless edge excitations, in a manner similar to that pointed out by Polychronakos [18]. Some of the novel properties of the new states can be understood by referring to the knowledge of the FQH fluid in condensed matter physics. We speculate that, upon including back-reactions of the giant gravitons, the FQH state would accordingly give rise to new geometries with singularities in IIB string theory, and that the novel properties associated with the FQH

state may help us understand how quantum effects resolve the singularities.

This paper is organized as follows. In Section II we briefly review the LLM's 1/2 BPS solutions in IIB supergravity, in particular the $AdS_5 \times S^5$ geometry from LLM solutions. We also review giant gravitons in $AdS_5 \times S^5$. Part of the purpose of this review is to set up the notations. In Section III we derive the effective theory which describes low energy dynamics in a special LLM geometry, the $AdS_5 \times S^5$ background, of giant gravitons moving in S^5. We shall show how this model reproduces the Landau problem for a single giant graviton. Then in Section IV, it is shown how to pass from the effective field theory for giant gravitons to a Chern-Simon finite-matrix model, which turns out to be essentially the same model proposed in ref. [18]. In Section V we shall demonstrate that the resulting matrix model for giant gravitons accommodates all essential features of FQHE, including the repulsive interaction

that favors the formation of the FQH state. And we speculate that novel properties of the FQH state may help us understand how quantum effects resolve the singularities in the new geometry that emerges when back-reactions are included. Finally, Section VI is devoted to a brief summary.

II.   1/2 BPS GEOMETRIES IN IIB THEORY AND GIANT GRAVITON PROBES

The class of 1/2 BPS states is known to play an important role in testing AdS/CFT correspondence. On the CFT side these states are associated to chiral primary operators with conformal weight $\Delta = J$, where J is a certain $U(1)$ charge in R-symmetry group. For small excitation energy $J \ll N$ (N being the rank of the gauge group), these BPS states on the AdS side correspond to graviton modes propagating in the bulk. When the excitation energy increases to $J \sim N$, some of these states are described by giant gravitons, i.e. spherical D3-branes either in the internal sphere[12] or in AdS[13, 14]. As excitation energy increases further the back-reaction cannot be ignored, and new geometries which preserve 16 supercharges are expected to emerge in IIB supergravity.

A.   LLM's IIB solutions

In a seminal paper[4], LLM explicitly derived the most general, smooth 1/2 BPS solutions of IIB supergravity, that are invariant under $SO(4) \times SO(4) \times R$ global symmetry. They contain the non-vanishing R-R 5-form flux of the form

$$F_{(5)} = F_{\mu\nu}dx^\mu \wedge dx^\nu \wedge d\Omega_3 + \tilde{F}_{\mu\nu}dx^\mu \wedge dx^\nu \wedge d\tilde{\Omega}_3, \tag{1}$$

where $\mu, \nu = 0, 1, 2, 3$, and $d\Omega_3$ and $d\tilde{\Omega}_3$ denote the volume forms of two three-spheres. The two $SO(4)$s are the rotational symmetries of these two three-spheres respectively. The dilaton-axion moduli is constant and the 3-form field strengths are set to be zero. Accordingly, LLM geometry is determined explicitly by the following metric and R-R 5-form measured in the unit $l_s = 1$,

$$
\begin{aligned}
ds^2 &= -h^{-2}(dt + V_i dx^i)^2 + h^2(dy^2 + dx^i dx^i) + ye^G d\Omega_3^2 + ye^{-G}d\tilde{\Omega}_3^2, \\
h^{-2} &= 2y\cosh G, \qquad\qquad z = \frac{1}{2}\tanh G, \\
y\partial_y V_i &= \epsilon_{ij}\partial_j z, \qquad\qquad y(\partial_i V_j - \partial_j V_i) = \epsilon_{ij}\partial_y z, \\
F_{(2)} &= F_{\mu\nu}dx^\mu \wedge dx^\nu = dB_t \wedge dt + dA_{(1)} = d(B_t dt + B_t V + \hat{B}), \\
\tilde{F}_{(2)} &= \tilde{F}_{\mu\nu}dx^\mu \wedge dx^\nu = d\tilde{B}_t \wedge dt + d\tilde{A}_{(1)} = d(\tilde{B}_t dt + \tilde{B}_t V + \hat{\tilde{B}}), \\
B_t &= -\frac{1}{4}y^2 e^{2G}, \qquad\qquad \tilde{B}_t = -\frac{1}{4}y^2 e^{-2G}, \\
d\hat{B} &= -\frac{1}{4}y^3 *_3 d\frac{z + 1/2}{y^2} \qquad d\hat{\tilde{B}} = -\frac{1}{4}y^3 *_3 d\frac{z - 1/2}{y^2},
\end{aligned}
\tag{2}
$$

where $i = 1, 2$, $t = x^0$, $y = x^3$. $*_3$ is the flat space epsilon symbol in the three dimensions parameterized by x_1, x_2, y. The full solution is determined by a single function $z(x_1, x_2, y)$, which obeys the Laplace equation in six dimensions

$$\partial_i\partial_i\frac{z}{y^2} + \frac{1}{y^3}\partial_y(y^3\partial_y\frac{z}{y^2}) = 0, \tag{3}$$

with y acting like the radial coordinate in four dimensions. This solution is non-singular as long as $z = \pm\frac{1}{2}$ on the two dimensional plane spanned by (x_1, x_2) at $y = 0$.

As long as $y > 0$, there are two 3-spheres S^3 and $\tilde{S}^3$, corresponding to the two $SO(4)$ isometries. At the $y = 0$ plane S^3 shrinks to a point in the region with $z = -\frac{1}{2}$, while $\tilde{S}^3$ shrinks in the region with $z = \frac{1}{2}$. Following the setup in Section 2.4 of ref.[4], we may choose a surface $\tilde{\Sigma}_2$ in the (y, x_1, x_2) space that ends at $y = 0$ on a closed non-intersecting curve

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lying in a region with $z = \frac{1}{2}$. A smooth five-manifold $\tilde{\Sigma}_5$ can be constructed by the fibration of $\tilde{S}^3$ over $\tilde{\Sigma}_2$. The five-form flux measured on this five-manifold is given by

$$\tilde{N} = -\mu_3 \int_{\tilde{\Sigma}_5} F_{(5)} = \frac{(\text{Area})_{z=-1/2}}{2\pi}, \tag{4}$$

where here the normalization $\mu_3 = T_3 = (2\pi)^{-3}$ in string units is used and $(\text{Area})_{z=-1/2}$ is the area of the region with $z = -1/2$ inside $\tilde{\Sigma}$ on $y = 0$ plane. We can as well construct another five-manifold Σ_5 by the fibration of S^3 over a surface Σ_2; Σ_2 ends in the region with $z = -\frac{1}{2}$ of the $y = 0$ plane. The five-form flux measured on Σ_5 is

$$N = \mu_3 \int_{\Sigma_5} F_{(5)} = \frac{(\text{Area})_{z=1/2}}{2\pi}. \tag{5}$$

Equations (4) and (5) indicate that the area of each connected region with either $z = \frac{1}{2}$ or $z = -\frac{1}{2}$ on the $y = 0$ plane is quantized. This is because the total 5-flux through any five-sphere is quantized by Dirac quantization condition.

The LLM solutions (1) and (2) have been shown to have the energy equal to the angular momentum and to preserve 16 supersymmetries, so they are indeed geometries satisfying the 1/2 BPS condition. They provide us a new theoretical setting to test the AdS/CFT duality at a new level.

B. $AdS_5 \times S^5$ from LLM

The LLM geometries are completely determined by the distribution of the sources for the function z on the boundary $y = 0$ plane. $AdS_5 \times S^5$ as the maximal BPS geometry is a special case of the LLM geometries, with the sources uniformly distributed in a disk with $z = -1/2$ on the $y = 0$ plane. Then the solution of the six dimensional Laplace equation (3) is given by

$$z(r, y; r_0) = \frac{r^2 - r_0^2 + y^2}{2\sqrt{(r^2 + r_0^2 + y^2)^2 - 4r^2 r_0^2}},$$
$$V_\phi(r, y; r_0) = \frac{1}{2} - \frac{r^2 + r_0^2 + y^2}{2\sqrt{(r^2 + r_0^2 + y^2)^2 - 4r^2 r_0^2}}, \tag{6}$$

where (r, ϕ) are the polar coordinates on the (x_1, x_2) plane, and r_0 is the radius of the disk. For this isotropic solution, $\hat{B}$ and $\tilde{\hat{B}}$ defined in (2) can be expressed as

$$\hat{B} = K(r, y; r_0) d\phi, \qquad\qquad \tilde{\hat{B}} = \tilde{K}(r, y; r_0) d\phi, \tag{7}$$

where K and $\tilde{K}$ are determined by the following differential equations

$$\partial_r K = -\frac{ry}{4}\partial_y z + \frac{r}{2}(z + \frac{1}{2}), \qquad \partial_y K = \frac{ry}{4}\partial_r z,$$

$$\partial_r \tilde{K} = -\frac{ry}{4}\partial_y z + \frac{r}{2}(z - \frac{1}{2}), \qquad \partial_y \tilde{K} = \frac{ry}{4}\partial_r z. \tag{8}$$

It is easy to check that the above equations are compatible with the Laplace equation (3).

Inserting (6) in the LLM solution (2) and performing the change of coordinate

$$y = r_0 \sinh\rho\sin\theta,$$

$$r = r_0 \cosh\rho\cos\theta, \tag{9}$$

$$\phi = \tilde{\phi} - t,$$

we obtain the standard $AdS_5 \times S^5$ metric in the global coordinates

$$ds^2 = r_0[-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega_3^2 + d\theta^2 + \cos^2\theta\, d\tilde{\phi}^2 + \sin^2\theta\, d\tilde{\Omega}_3^2]. \tag{10}$$

Matching it with $AdS_5 \times S^5$ geometry from the near-horizon limit of D3-branes, we have $r_0 = R^2_{AdS} = R^2_S = \sqrt{4\pi g_s \tilde{N}}$. Meanwhile, from eq. (9) we see that the interior of the disk on the $y = 0$ plane corresponds to $\rho = 0$, i.e., the center of AdS_5, while the exterior of the disk on the $y = 0$ plane corresponds to $\theta = 0$, i.e., the north-pole of S^5.

The last equation in (9) is linear in time. This implies that the LLM coordinates and the global coordinates of $AdS_5 \times S^5$ differ from each other by a relative frame rotation in the (x_1, x_2) plane. In the LLM frame, the $AdS_5 \times S^5$ metric (10) is seen by an observer who co-moves with a rotating frame in the (x_1, x_2) plane. Conversely, in the global frame for $AdS_5 \times S^5$, an observer who moves along the $\tilde{\phi}$-circle in S^5 will see the LLM metric (6) determined by a disk source. So the double scaling limit of large angular momentum and large radius of $\tilde{\phi}$-circle is just the PP-wave limit of $AdS_5 \times S^5$ [15].

For convenience for the following discussions, we will evaluate the $y \to 0$ limit of various functions. For the interior of the disk, $(r^2 < r_0^2)$, we have

$$z \to -1/2, \qquad\qquad V_\phi \to \frac{r^2}{r^2 - r_0^2}, \tag{11}$$

$$\begin{cases} B_t \to 0, \\ \tilde{B}_t \to -\frac{(r^2-r_0^2)^2}{4r_0^2}, \end{cases} \qquad\qquad \begin{cases} K \to 0, \\ \tilde{K} \to -r^2/4 \end{cases}$$

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while for the exterior of disk $(r^2 > r_0^2)$,

$$z \to 1/2, \qquad\qquad V_\phi \to -\frac{r_0^2}{r^2 - r_0^2}, \qquad (12)$$

$$\begin{cases} B_t \to -\frac{(r^2-r_0^2)^2}{4r_0^2}, \\ \tilde{B}_t \to 0, \end{cases} \qquad\qquad \begin{cases} K \to r^2/4, \\ \tilde{K} \to 0. \end{cases}$$

We see that the radii of the two three-spheres, S^3 inside AdS_5 and $\tilde{S}^3$ inside S^5, are proportional to $\sqrt{|B_t|}$ and $\sqrt{|\tilde{B}_t|}$, respectively. Thus in the interior (exterior) of the disk at $y = 0$, S^3 ($\tilde{S}^3$) shrinks to zero size.

C. Giant graviton revisited

In this subsection, we will revisit the giant graviton in $AdS_5 \times S^5$[12], but use the LLM's expressions in which the $R - R$ field strength is slightly different from that given in refs. [12, 13]. This re-formulation will be helpful for the discussions in section V.

A giant graviton is a spherical D3-brane, which has the same quantum numbers as a graviton. So the D3-brane wraps a 3-sphere either in AdS_5 or in S^5. Its dynamics is governed by the world-volume action

$$S = -T_3 \int d^4\xi \sqrt{-\det(P[G]_{ab})} + T_3 \int P[C_{(4)}]. \qquad (13)$$

where $G_{\mu\nu}$ and $C_{(n)}$ are the background metric and R-R n-form potential, respectively. $P[\cdots]$ denotes the pullback of the enclosed spacetime tensor to the brane world-volume. For our purpose, we only consider D3-branes wrapping on a 3-sphere $\tilde{S}^3$ in S^5. Then we may take the static gauge:

$$\xi^0 = t, \ \xi^m = \omega^m \qquad (m = 1, 2, 3), \qquad (14)$$

with ω^m being the coordinates of $\tilde{S}^3$, and consider a trial solution of the form, in the polar coordinates on the (x_1, x_2) plane,

$$r = \tilde{r}r_0 \qquad (0 < \tilde{r} = \text{const.} \le 1), \qquad y = 0, \qquad \phi = \tilde{\phi}(t) - t. \qquad (15)$$

Substituting the background metric (10), the R-R potential (1) and the ansatz (15) into the world-volume action (13) and integrating over the angular coordinates yield the following Lagrangian

$$L(t)/r_0^2 = -(1 - \tilde{r}^2)^{3/2}\sqrt{1 - \tilde{r}^2\dot{\tilde{\phi}}^2} + \tilde{r}^4\dot{\tilde{\phi}} + (1 - 2\tilde{r}^2). \qquad (16)$$

The corresponding Hamiltonian and the angular momentum J conjugate to $\tilde{\phi}$ read

$$H/r_0^2 = \sqrt{(1 - \tilde{r}^2)^3 + \frac{1}{\tilde{r}^2}(j - \tilde{r}^4)^2} - (1 - 2\tilde{r}^2),$$

$$j = \frac{J}{r_0^2} = \frac{(1 - \tilde{r}^2)^{3/2} \tilde{r}^2 \dot{\tilde{\phi}}}{\sqrt{1 - \tilde{r}^2 \dot{\tilde{\phi}}^2}} - \tilde{r}^4. \tag{17}$$

The above Hamiltonian possesses two local minima. One of them is at $\tilde{r} = 1$, the boundary of the disk, where the D3-brane shrinks to point and behaves like an ordinary graviton. Another minimum locates at $\tilde{r} = \sqrt{j}$ for fixed $j \in [0, 1)$. D3-branes at this minimum preserve non-vanishing size, and are called "giant gravitons" in the literature. The energy of those classical stable "giant gravitons" is given by

$$E_g = J = \tilde{r}^2 r_0^2 = r^2. \tag{18}$$

This is precisely the 1/2 BPS condition.

It is interesting to note that one always has $\dot{\tilde{\phi}} = 1$ at these minima, independent of J. For an observer co-moving with the giant graviton, he/she sees that the world-volume dynamics along the circle is labeled by the variable $\varphi = \tilde{\phi} - t$. In other words, the Lagrangian (16) should be rewritten as

$$L(t) = \frac{1}{2} r^2 \dot{\varphi}^2 + r^2 \dot{\varphi} + \cdots. \tag{19}$$

It is obvious that a non-zero $\dot{\varphi}$ implies quantum fluctuations of a giant graviton around its classical stationary points.

III. 1/2 BPS DYNAMICS OF GIANT GRAVITONS IN LLM GEOMETRY

Now let us add a few giant gravitons in the LLM geometry. Suppose the number of giant gravitons, M, is much less than the number of background five-form flux $\tilde{N}$, so that the back-reaction of the giant gravitons on background geometry can be ignored. They are assumed to be located at $\rho = 0$ (hence inside the $z = -1/2$ disk on the $y = 0$ plane) and to wrap $\tilde{S}^3$ in S^5. We shall show that the background R-R flux provides a homogeneous magnetic field perpendicular to the $y = 0$ plane and the dynamics of giant graviton in the (x_1, x_2) plane at $y = 0$ is the same as that of electrons moving in a 2d plane with a constant transversal magnetic field.

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The low energy dynamics of giant gravitons in a fixed background (without NS B-field) is known [16] to be described by the following non-Abelian generalization of the DBI action (13)

$$S_{DBI} = -T_3 \int d^4\xi \, \mathrm{STr}\sqrt{-\det(P[G_{ab} + G_{ai}(Q^{-1} - \delta)^{ij}G_{jb}] + 2\pi F_{ab})\det(Q^i{}_j)}, \qquad (20)$$

with[1]

$$Q^i{}_j = \delta^i{}_j + i[X^i, X^k]G_{kj}, \qquad (21)$$

plus the Chern-Simons action [17], with the bosonic part

$$S_{CS} = T_3 \int \mathrm{STr}\left(P[e^{i i_X i_X}(\sum_n C^{(n)})]e^{2\pi F} \right), \qquad (22)$$

supplemented by proper fermionic terms . Here the transverse coordinates X^i, as world-volume scalar fields, are $M \times M$ matrices in the adjoint representation of the $U(M)$ gauge group. $\mathrm{STr}(\cdots)$ denotes a symmetrical trace in gauge

group. The operator i_X is the contraction with X_i:

$$i_X i_X C^{(n)} = \frac{1}{(n-2)!} X^{i_1} X^{i_2} C^{(n)}_{i_1 i_2 i_3 \cdots i_n} dx^{i_3} \wedge \cdots \wedge dx^{i_n}. \qquad (23)$$

Now we consider the giant gravitons locating at the center of AdS ($\rho = 0 \Rightarrow y = 0$) and, as D3-branes, all wrapping on the same $\tilde{S}^3$ in S^5. We take the static gauge (14) again for the world-volume coordinates. Recall that our purpose is to study the 1/2 BPS sector in AdS/CFT duality, treated as a closed sector in view of the arguments given in ref. [6]. Thus we are allowed to truncate the degrees of freedom of the giant gravitons to those in the 1/2 BPS sector: Namely the world-volume gauge field is taken to be zero, and we turn on the transverse scalar fields along x_1, x_2 directions only. (According to the arguments in ref. [6], there exists a limit in which all other degrees of freedom are decoupled, i.e. they cost too much energy and therefore can be integrated out at low energy.) Thus by inserting the background metric and five-form field strength given by (1), (2) and (6) into (20) and (22), we obtain

$$S_{BI} = -2\pi^2 T_3 \int dt \, \mathrm{Tr}\{4|\tilde{B}_t| + 4|\tilde{B}_t|V_i\dot{X}_i - \frac{1}{2}\dot{X}_i^2 + \frac{1}{2}[X_1, X_2]^2\} + \cdots, \qquad (24)$$

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[1] We use i, j, k to denote the directions transverse to the branes, and a, b, c those along the branes.

and

$$S_{CS} = -8\pi^2 T_3 \int dt \, \text{Tr}\{\tilde{B}_t + (\tilde{B}_t V_i + \hat{\tilde{B}}_i)\dot{X}_i\} + \cdots. \tag{25}$$

Here X_i's have been restricted to the lowest order modes in the expansion of spheric harmonics on $\tilde{S}^3$, because we focus on the 1/2 BPS states. The dots "$\cdots$" denotes higher order terms in powers of M/N that are suppressed in our probe approximation.

The quartic term plays no role in the 1/2 BPS sector. Using eqs. (11) and (12) in (24) and (25), we get a matrix model with the following Lagrangian:

$$L(t) = \frac{1}{2l_s}\text{Tr}\dot{X}_i^2 + \frac{1}{l_s^2}\epsilon_{ij}\text{Tr}X^i\dot{X}^j, \tag{26}$$

where we have restored the l_s-dependence and rescaled X_i by $X_i \to \sqrt{4\pi}X_i$.

For a single giant graviton, with $M = 1$, X_1 and X_2 become real numbers; then in polar coordinates and with $\rho = 0$, the above Lagrangian is reduced to eq. (19). This consistency implies that the Lagrangian (26) describes the quantum fluctuations of the giant graviton around their classical stationary point in the 1/2 BPS sector. On the other hand, the Lagrangian (26) with $M = 1$ precisely describes the Landau problem, for a non-relativistic charged particle with mass $m = 1/l_s$ moving in a constant magnetic field $B = 2/l_s^2$.

For the number of giant gravitons $M > 1$, X_i's are $M \times M$ matrices. Their diagonal elements describe the positions of the giant gravitons, while the off-diagonal elements describe the interactions between the giant gravitons. At low energy, one may take the limit $l_s \to 0$. This is equivalent to take the limit $B/m \to \infty$, i.e. the cyclotron frequency in the Landau problem tends to infinity. However, for a large number of giant gravitons the classical Lagrangian (26) does not provide a proper description for the quantum giant-graviton fluid, since the 2-plane should become a non-commutative 2-plane when all giant gravitons are projected down to the lowest Landau levels (LLL). In the next section, we shall show that the matrix model (26) has to be supplemented by a non-commutativity constraint in an incompressible fluid phase, so that the quantum behavior of the many giant graviton system is that of a quantum Hall fluid, described by a Chern-Simons matrix model.

To conclude this section, we note that the Lagrangian (26) can be recast into a manifestly $U(M)$ gauge invariant form:

$$L(t) = \frac{m}{2}\text{Tr}(DX_i)^2 + \frac{B}{2}\epsilon_{ij}\text{Tr}(X^i DX^j), \tag{27}$$

where $DX_i = \dot{X}_i + i[A_0, X_i]$ with A_0 the electric potential.

IV. CHERN-SIMONS FINITE-MATRIX MODEL FOR GIANT GRAVITONS

Since the essential non-commutativity condition is absent, the matrix model (27) is not sufficient to describe a QH fluid. We shall show that this condition naturally emerges as soon as the quantum physics of giant gravitons is considered.

A. Symmetry breaking

Since the Lagrangian (27) is invariant under $U(M)$ gauge symmetry, one of the Xs, e.g. X_1, can be diagonalized by a gauge transformation[2]. In this eigenvalue basis, with notation $(X_1)_{mn} = \delta_{mn} x_{1m}, (X_2)_{mn} = y_{mn}, y_{nn} = x_{2n}$, a typical classical Lagrangian of the matrix model reads

$$L = \frac{m}{2} \sum_{i,n} \dot{x}_{in}^2 + \frac{m}{2} \sum_{m \neq n} \dot{\bar{y}}_{mn} \dot{y}_{mn} + \frac{B}{2} \sum_{i,n} \epsilon_{ij} x_n^i \dot{x}_n^j - U(x,y). \tag{28}$$

However, quantum mechanically there is a non-trivial change of measure in path integral from the matrix-element basis to the eigenvalue basis. So the Hamiltonian in the quantum theory is given by

$$
\begin{aligned}
H_q &= -\frac{1}{2m} \sum_{i,n} \frac{1}{\Delta^2} (\frac{\partial}{\partial x_{in}} - iB\epsilon_{ij}x_j)\Delta^2(\frac{\partial}{\partial x_{in}} - iB\epsilon_{ij}x_j) - \frac{1}{2m} \sum_{m,n} \frac{\partial}{\partial \bar{y}_{mn}} \frac{\partial}{\partial y_{mn}} + U(x,y) \\
&= \frac{1}{\Delta(x_1)} \tilde{H} \Delta(x_1),
\end{aligned}
\tag{29}
$$

where $\Delta(x) = \prod_{n<m}(x_n - x_m)$ is the Van der Monde determinant, and $\tilde{H}$ is the Hamiltonian corresponding to the Lagrangian (28).

The eigenfunction $\tilde{\psi}$ of $\tilde{H}$ is related to an eigenfunction ψ of H_q by $\tilde{\psi}(x,y) = \Delta(x_1)\psi(x,y)$. It implies that the Hamiltonian (28) in the eigenvalue basis describes a quantum system of particles obeying Fermi statistics[3] from the requirement that any pair of x_{1n} are different. In other words, we may have two perspectives on the physics of giant gravitons at the quantum level. One is to use the Hamiltonian (29) of $U(M)$ gauge symmetry to describe the quantum physics, with M giant gravitons understood semiclassically coinciding. Another perspective

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[2] Here we do not adopt the argument of using a *complexified* gauge group to diagonalize X_1 and X_2 simultaneously.
[3] This is a 1-d fermion system instead of 2-d one, unlike that extracted from $\mathcal{N} = 4$ SYM[6]. It indeed makes sense at $m/B \to 0$ because the phase space is two dimensional at this limit.

is to just use the Hamiltonian obtained from the classical Lagrangian (28) to describe the quantum fluctuations of giant gravitons, leaving M giant gravitons separated from each other on the (x_1, x_2)-plane. In the latter case the world-volume gauge symmetry is broken from $U(M)$ to $U(1)^M$. So the degrees of freedom corresponding to the off-diagonal elements of X_2, which give excitation modes of open string stretched between different giant gravitons, become heavy and are frozen out at the low energy limit $l_s \to 0$. In continuous fluid description, two different perspectives were referred as two different fluids, the string fluid and brane fluid[19]. They should be related to each other, like the correspondence between matrix basis and eigenvalue basis in the toy matrix model discussed by Berenstein[6]. For our purpose, the latter perspective is more convenient. Thus we end up with the following Lagrangian from Eq. (27):

$$L = \sum_n \left[\frac{m}{2} \dot{x}_{in}^2 + \frac{B}{2} \epsilon_{ij} x_n^i \dot{x}_n^j - V(x_{in}) \right], \tag{30}$$

supplemented with Fermi statistics condition. Here we have included a potential $V(x_{in})$, which is assumed to arise from short-range interactions and will be shown in the next section to naturally emerge when we choose a correct vacuum. The Lagrangian (30) describes a collection of non-relativistic charged fermions moving in a plane subject to a perpendicular magnetic field. In the $m/B \to 0$ limit, all particles are projected into the LLL, which is infinitely degenerated. The degenerated states in the LLL are distinguished by the quantum number of angular momentum j. Due to Fermi statistics, different giant gravitons occupy states with different j.

B.   Incompressible Fluid description and non-commutative Chern-Simons theory[4]

Following refs. [3, 20], we treat the long distance behavior of the above system as a dissipationelss fluid. The discrete label n is replaced by a pair of continuous coordinates (y_1, y_2); these coordinates are co-moving coordinates subject to the condition that the density of particle number, ρ_0, is constant. Accordingly, the Lagrangian (30) can be rewritten as

$$L(t) = \int d^2 y \, \rho_0 \left[\frac{m}{2} \dot{x}_i^2(y) + \frac{B}{2} \epsilon_{ij} x_i(y) \dot{x}_j(y) - V(\rho_0 | \frac{\partial y}{\partial x} |) \right]. \tag{31}$$

[4] This subsection does not contain new results. It just follows the treatment in [3, 20], to show how non-commutativity comes about in our matrix model. The readers familiar with this issue may skip this subsection.

The potential V plays a role when distances among giant gravitons are small, i.e., it arises out of short-range forces. It leads to an equilibrium when the real space density is ρ_0. The fact that such an equilibrium state reaches the minimal energy is essentially an assumption, which may not be true if the giant gravitons are in, say, a Wigner crystal state. In the following we shall always make this assumption as a working hypothesis. Thus, the Jacobian $|\partial y/\partial x|$ equals to 1 in equilibrium.

The Lagrangian (31) is invariant under the area preserving diffeomorphisms (APD) of y-plane, generated by the infinitesimal transformations

$$\delta y_i = \epsilon_{ij}\frac{\partial \alpha(y)}{\partial y_j}, \quad \Rightarrow \quad \delta x_i = \epsilon_{jk}\frac{\partial x_i}{\partial y_j}\frac{\partial \alpha(y)}{\partial y_k}. \tag{32}$$

Small deviations from the equilibrium can be parameterized by a vector field A_i, defined by

$$x_i = y_i + \epsilon_{ij}\theta A_j, \tag{33}$$

where $\theta \ll 1$ is a convenient parameter to control the expansion. The APD (32) leads to the following transformation for A:

$$\delta A_i = \frac{\partial \tilde{\alpha}}{\partial y_i} + \epsilon_{lm}\theta\frac{\partial A_i}{\partial y_l}\frac{\partial \tilde{\alpha}}{\partial y_m}, \tag{34}$$

where $\tilde{\alpha} = \alpha/\theta$. The first term is in the standard form of an Abelian gauge transformation.

Moreover, the APD invariance implies that there exists a conserved charge, given by

$$\int d^2y \Pi_i \delta x_i. \tag{35}$$

In the limit $m/B \to 0$, the conjugate momentum density $\Pi_i \propto \epsilon_{ij}x^j$. Then the conserved current is the Jacobian from x to y. Thus, the equation of motion is supplemented with the constraint

$$|\frac{\partial x}{\partial y}| = \frac{1}{2}\epsilon_{ij}\epsilon_{mn}\frac{\partial x_m}{\partial y_i}\frac{\partial x_n}{\partial y_j} = 1. \tag{36}$$

Namely the fluid density in x-space is constant, or the fluid is incompressible, since by definition the fluid density is constant in y-coordinates. This constraint can be viewed as a Gauss law constraint and can be added into the action via a Lagrangian multiplier A_0. Then in the limit $m/B \to 0$, the Lagrangian reads

$$L(t) = \int d^2y \, \rho_0 \left[\frac{B}{2}\epsilon_{ij}(\dot{x}^i - \theta\{x_i, A_0\})x^j + \theta A_0 - V(\rho_0|\frac{\partial y}{\partial x}|)\right], \tag{37}$$

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where the Poisson bracket is defined by

$$\{F(y), G(y)\} = \epsilon_{ij}\partial_i F \partial_j G. \tag{38}$$

Eqs. (34) and (38) exhibit the structure of non-commutative $U(1)$ gauge theory with spatial-spatial non-commutativity: To the first order in θ expansion, eq. (34) is nothing but non-commutative gauge transformation, while eq. (38) is the non-commutative commutator. This observation led Susskind to propose [3] that, beyond the linear order, the APD invariance requires the action to be the non-commutative Chern-Simons (NCCS) theory (with $\nu = 1/B\theta$)

$$L_{NCCS} = \frac{1}{4\pi\nu}\epsilon^{\mu\nu\rho}\left(A_\mu \star \partial_\nu A_\rho + \frac{2i}{3}A_\mu \star A_\nu \star A_\rho\right), \tag{39}$$

and that (39) provides the correct framework for the quantum Hall system with ν being the filling fraction. The NCCS action (39) can be obtained from a non-commutative matrix mechanics. The NC matrix mechanics is given by

$$L = \frac{B}{2}\epsilon_{ij}\text{Tr}(\dot{X}_i + i[A_0, X_i])X_j + B\theta\text{Tr}A_0, \tag{40}$$

with $X_i = y_i + \epsilon_{ij}\theta A_j$. Here the constant matrices y_i are chosen to satisfy $[y_i, y_j] = i\theta\epsilon_{ij}$.

Collecting the above ingredients together, we conclude that one should add a term $B\theta\text{Tr}A_0$ into our matrix model (27) in the large B limit. This term yields a non-commutativity constraint, and compensates the anomaly arising from the change of measure in the large B limit. The matrix Lagrangian (27) plus the additional term $B\theta\text{Tr}A_0$ will be shown to describe a quantum Hall system of giant gravitons. In this way, we have demonstrated that the 1/2 BPS dynamics of giant gravitons in the LLM geometry background is essentially that of a quantum Hall system.

C. Edge excitations and a finite-matrix model for QHE

The Gauss law constraint that follows from the Lagrangian (40) implies the non-commutativity

$$[X_1, X_2] = i\theta, \tag{41}$$

which can be solved only with infinite matrices. For finite matrices of order M, this constraint can be satisfied only up to order $1/M$. To obtain a finite-matrix model with the

Gauss law exactly satisfied, one needs to add extra degrees of freedom. In our case, we argue this necessity as follows. Since different giant gravitons occupy different states labelled by angular momentum j of the LLL, the distance between two nearest giant gravitons is of order $\sqrt{j/B} - \sqrt{(j-1)/B} \sim l_s/\sqrt{j}$. When j is large enough, open string excitations stretched between the nearest giant gravitons may become very light and can be excited at low energy. Obviously, these almost gapless excitations prefer j as large as possible, namely in the region close to the boundary of the quantum Hall droplet. In the fluid description, we may introduce a boundary Lagrangian to describe these excitations:

$$L_b = \int d^2 y \rho_0 \delta(\Gamma(y_1, y_2)) \left[\frac{m_b}{2} \dot{\phi}^*(y)\dot{\phi}(y) + \frac{i}{2} B(\dot{\phi}^*\phi - \phi^*\dot{\phi}) - \frac{B\mu_b}{2}\phi^*(y)\phi(y) \right], \qquad (42)$$

where $\Gamma(y_1, y_2) = 0$ defines the boundary, $\phi(y)$ is a complex field defined on the boundary only and $\mu_b \sim 1/(Ml_s)$. In fact, there are apparently two boundary fields and, later, we will show that they correspond to the continuum limit of the off-diagonal elements $(X_i)_{1n}$ and $(X_i)_{n1}$ or their complex combination

$$\Psi_{n1} = (X_1)_{n1} + i(X_2)_{n1}, \ \ \Psi_{1n} = (X_1)_{1n} + i(X_2)_{1n}. \qquad (43)$$

(Here the convention is that the giant graviton with the largest j is labelled by index "1"). However, half of them can always be gauged away by the residual $U(1)^M$ gauge symmetry, and only one boundary field is physical. (This corresponds to the well-known fact that the edge states of a quantum Hall droplet are chiral, in the sense that edge wave propagates along the boundary only in one direction, not in the opposite direction.)

Under an infinitesimal APD, $\phi(y)$ transforms as (32). But the conserved charge now is given by

$$\int d^2 y \ \rho_0 \Pi_i \delta x_i + \int d^2 y \rho_0 \delta(\Gamma(y_1, y_2))(\Pi_\phi \delta\phi + \Pi_{\phi^*}\delta\phi^*). \qquad (44)$$

For small B, it implies a vortex-like excitation at the boundary. In the large-B limit, it changes the constraint (36) to

$$\frac{1}{2}\epsilon_{ab}\{x_a, x_b\} - 2i\{\phi^*, \phi\}\delta(\Gamma) = 1. \qquad (45)$$

Thus, with a large B, we get a Lagrangian to describe this fluid with edge excitations:

$$\begin{aligned} L(t) &= \int d^2 y \ \rho_0 \left[\frac{B}{2}\epsilon_{ij}(\dot{x}^i - \theta\{x_i, A_0\})x^j + \theta A_0 - V(\rho_0|\frac{\partial y}{\partial x}|) \right] \\ &\quad - \int d^2 y \ \rho_0 \delta(\Gamma) \left[iB\phi^*(\dot{\phi} - \theta\{A_0, \phi\}) + \frac{B\mu_b}{2}\phi^*\phi \right]. \end{aligned} \qquad (46)$$

It has been mentioned in the previous subsection that $\{A, B\}$ is the first order truncation of the non-commutative commutator $[A, B]_\star = A \star B - B \star A$ defined by star product, and the full expression is assumed to be the result of the substitution of Poisson brackets by the non-commutative commutators. Following the same logic, we pass from the above Lagrangian to the matrix description by the replacements

$$\int d^2y \, \rho_0 \to \text{Tr}, \qquad \qquad \theta\{A, B\} \to [A, B],$$
$$\int d^2y \, \rho_0\delta(\Gamma)A \to A_{11}, \tag{47}$$

where again the index "1" is referred to the giant graviton with the maximal angular momentum j in LLL. In particular, one has

$$\theta \int d^2y \, \rho_0\delta(\Gamma)\phi^*\{A_0, \phi\} \to \Psi^\dagger_{1m}(A_0)_{mn}\Psi_{n1} - \Psi_{1m}(A_0)_{mn}\Psi^\dagger_{n1}. \tag{48}$$

We have mentioned that half of degrees of freedom in Ψ_{1m} and Ψ_{m1} may be gauged away. Finally we obtain a matrix model described by the following $U(M)$ invariant Lagrangian:

$$L = \frac{B}{2}\text{Tr}(\epsilon_{ij}X^i DX^j + \theta A_0) + B\Psi^\dagger(iD\Psi - \mu_b\Psi), \tag{49}$$

where Ψ transforms as fundamental representation under $U(M)$. In the above Lagrangian we have taken the limit $m/B \to 0$.

The matrix model presented in (49) is slightly different from Polychronakos's finite matrix model[18] by the absence of a confining X^2 potential. However, eq. (18) indicates that the giant gravitons are actually confined around the origin to minimalize their energy. It means that the matrix Lagrangian (49), though capturing the most important features of the giant graviton system, does not accommodate all information of the system. To be precise, the Hamiltonian, H', obtained from the matrix Lagrangian (49) describes only the quantum fluctuations of giant gravitons around the classical stationary point or the corresponding 1/2 BPS geometries/states. In other words, we should identify $H' = H - H_0$, where $H_0 = J$ is the energy of 1/2 BPS states. Because the angular momentum J should be treated quantum mechanically, the system must be quantized with the full Hamiltonian H, instead of merely H'. Recalling that $H' = H_\Psi = B\mu_b\Psi^\dagger\Psi$, supplemented with the Gauss law constraint, the full Hamiltonian is of the following form:

$$H = J + H_\Psi = \frac{B}{2m}\text{Tr}(X_1\frac{\partial L}{\partial\dot{X}_2} - X_2\frac{\partial L}{\partial\dot{X}_1}) + H_\Psi = \frac{B^2}{4m}\text{Tr}(X_1^2 + X_2^2) + H_\Psi. \tag{50}$$

The X-part of the above Hamiltonian behaves as M^2 harmonic oscillators in the matrix basis, since X_1 and X_2 are conjugated each other.

V.   GIANT GRAVITON FLUID AS FRACTIONAL QUANTUM HALL FLUID

In this section we shall quantize the finite-matrix Hamiltonian (50) supplemented with Gauss law constraints derived from from the Lagrangian (49), to show that indeed the 1/2 BPS dynamics of giant gravitons in $AdS_5 \times S^5$ allows the formation of fractional quantum Hall (FQH) fluids with filling factor $\nu = 1/k$, with k a positive odd number. There are several approaches to quantize this model: (1) Canonical approach with Gauss' constraint; (2) Path integral approach; (3) Reduced canonical approach (first to solve the classical constraints and then to quantize). Though they all lead to essentially the same physics, each approach has its own advantages in revealing some aspects of the underlying physics.

A.   "Classical" quantum Hall droplet

The Gauss law constraint derived from the Lagrangian (49),

$$G = -i[X_1, X_2] + \Psi\Psi^\dagger - \theta = \frac{1}{2}[A, A^\dagger] + \Psi\Psi^\dagger - \theta = 0, \tag{51}$$

implies that the energy is discrete, even at the classical level, with an energy gap of order θ. To reveal this, we need to solve the above constraint as well as the classical equations of motion. Classically, inserting the solution of equation of motion,

$$X_1 + iX_2 = A, \qquad \Psi = e^{-i\mu_b t}\sqrt{M\theta}|v\rangle, \tag{52}$$

with A a constant $M \times M$ matrix and $|v\rangle$ a constant vector of unit length, into the Hamiltonian (50), we obtain

$$H = \frac{B^2}{4m}\mathrm{Tr}A^\dagger A + MB\theta\mu_b. \tag{53}$$

In the oscillator basis, we may choose $|v\rangle = |M - 1\rangle$. The traceless part of the constraint (51) has many solutions for A. Each of these solutions corresponds to a different distribution of the giant gravitons on the 2-plane. For example, the solution corresponding to the ground state is

$$A = \sqrt{2\theta}\sum_{n=1}^{M-1}\sqrt{n}|n - 1\rangle\langle n|. \tag{54}$$

It yields the radius squared matrix:

$$R^2 = X_1^2 + X_2^2 = \frac{1}{2}(A^\dagger A + AA^\dagger)$$

$$= \sum_{n=0}^{M-2} \theta(2n+1)|n\rangle\langle n| + \theta(M-1)|M-1\rangle\langle M-1|. \tag{55}$$

Semi-classically the M giant gravitons in this case are uniformly distributed in the (x_1, x_2) plane on a disk with radius $\sqrt{2M\theta}$, (see Fig. 1a), with density $\rho_{gg} = 1/(2\pi\theta)$ and the inverse filling factor $\nu^{-1} = B/(2\pi\rho_{gg}) = B\theta$ in the analogy to the QH "droplet". Later we will show that this droplet can generally be in a FQH liquid phase when the density is at the right value. The total classical energy of this system is

$$E = \frac{B^2}{4m}\theta M(M-1) + MB\theta\mu_b.$$

The classical energy contributed by the edge states is order $M\mu_b \sim 1/l_s$, and can be ignored compared with the energy contributed by X_i. This is consistent with the known fact that the edge excitations are essentially gapless.
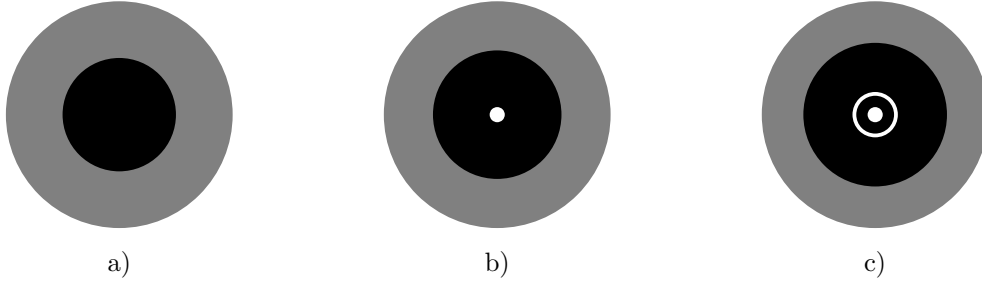

a)   b)   c)

FIG. 1: The giant gravitons in $AdS_5 \times S^5$ and its analogy to the QH "droplet". The gray region denotes the $AdS_5 \times S^5$ background. The giant gravitons are uniformly distributed on the black region and the quasi-hole excitations in this giant graviton background are denoted by the white region. a) The (fractional) QH 'droplet" formed from the condensation of the giant gravitons. b) A quasi-hole excitation at origin. c) Several quasi-hole excitations.

Another interesting special solution is

$$A = \sqrt{2\theta}\left(\sqrt{q}|M-1\rangle\langle 0| + \sum_{n=1}^{M-1}\sqrt{n+q}|n-1\rangle\langle n|\right), \tag{56}$$

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where $q > 0$. It is known to be a quasi-hole excitation with charge $-q$ (a defect) at the origin in the FQH background (Fig. 1b). It increases the radius of the state $|m\rangle$ from $\sqrt{(2m+1)\theta}$ to $\sqrt{(2m+2q+1)\theta}$. This corresponds to the excitation of the giant graviton at the $j = 0$ state (with j the angular momentum in LLL) to the $j = 1$ state, while the giant graviton at the $j = \ell$ state is excited to the $j = \ell + 1$ state, etc. The total number of giant gravitons is unchanged. Finally, a rather general solution can be constructed as follows:

$$A = \sqrt{2\theta} \sum_{i=1}^{m} \left(\sqrt{q_i}|n_i\rangle\langle n_{i-1}| + \sum_{n=n_{i-1}+1}^{n_i} \sqrt{n+q_i}|n-1\rangle\langle n| \right), \tag{57}$$

where $|n_0\rangle = |0\rangle$, $|n_m\rangle = |M-1\rangle$ and $m \ll M$. This solution can interpreted as several quasi-hole excitations with charges $-q_i$, at different radii in the QH background (Fig.1c). Each of these solutions corresponds to a classical, stable configuration of the giant gravitons.

Finally, we remark that both the classical Hamiltonian and the angular momentum depend on the trace of the radius-squared matrix only. Therefore, both the energy and angular momentum are quantized, even at the classical level.

B. Gauss' law in the canonical approach

To quantize the matrix model (49), we treat the matrix elements of X_1 and X_2 as operators, and impose the canonical commutation relations

$$[(\mathbf{X}_1)_{mn}, (\mathbf{X}_2)_{rs}] = \frac{i}{B}\delta_{mr}\delta_{ns}, \tag{58}$$

because the Lagrangian is first order in time derivative. Hereafter we shall use the boldface letters to denote operators in the quantum theory. The Hamiltonian for the $\mathbf{X}$'s is so ordered as to be given by $(\mathbf{Z} = \mathbf{X}_1 + i\mathbf{X}_2)$

$$\mathbf{H}_X = \frac{B^2}{4m} \sum_{mn} (\mathbf{Z}^\dagger_{mn}\mathbf{Z}_{mn} + \frac{1}{B}), \tag{59}$$

corresponding to M^2 harmonic oscillators. The components of the column vector $\mathbf{\Psi}$ correspond to M oscillators and are quantized to be bosons with the commutation relations

$$[\mathbf{\Psi}_m, \mathbf{\Psi}^\dagger_n] = \frac{1}{B}\delta_{mn}. \tag{60}$$

Further the operator ordering ambiguity in the quantum version of the classical Gauss' law constraint (51) is fixed by requiring that as quantum generators of unitary rotations

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of both $\mathbf{X}_1$, $\mathbf{X}_2$ and $\boldsymbol{\Psi}$, the operator $\mathbf{G}$ should satisfy the commutation relations of the $U(M)$ algebra. Therefore, the traceless ($SU(M)$) part of the Gauss' law operator $\mathbf{G}$ can be constructed as the sum of two terms, $\mathbf{G}_X$ and $\mathbf{G}_{\Psi}$, which are the well-known bilinear realization of the $SU(M)$ algebra in the Fock basis of bosonic oscillators in the adjoint and fundamental representations, respectively.

$$\mathbf{G} \equiv \mathbf{G}_X + \mathbf{G}_{\Psi}$$
$$= -i\Big(\mathbf{Z}^{\dagger}_{mk}\mathbf{Z}_{nk} - \mathbf{Z}^{\dagger}_{nk}\mathbf{Z}_{mk}\Big)E_{mn} + \boldsymbol{\Psi}^{\dagger}_m T^a_{mn} \boldsymbol{\Psi}_n T^a \tag{61}$$

where E_{mn} is the matrix with only the element at the m-th row and n-th column being one and all other elements zero; T^a is the matrix for the generator of $SU(M)$ in the fundamental representation. Then the traceless part of the quantum Gauss's law is given by

$$(\mathbf{G}^a_X + \mathbf{G}^a_{\Psi})|phys> = 0. \tag{62}$$

The $U(1)$ part of the quantum Gauss's law, similar to that in quantum electrodynamics, just requires the total $U(1)$ charge of the model to vanish:

$$(\boldsymbol{\Psi}^{\dagger}_m \boldsymbol{\Psi}_m - M\theta)|phys> = 0. \tag{63}$$

The two constraints (62) and (63) together require the physical states to be $U(M)$ singlets.

C. Quantization of the inverse filling factor

Because of purely group-theoretical reasons, Gauss' law constraints (62) and (63) impose a severe restriction on the possible value for the parameter $B\theta$, which is closely related to the inverse filling factor ν^{-1} for giant gravitons. Eq. (62) requires that the physical states are invariant under $SU(M)$. Consequently, the representation of $\mathbf{G}_X$ and that of $\mathbf{G}_{\Psi}$ must be conjugate to each other, so that it is possible to form an $SU(M)$ singlet. The X-part is known to be formed by the tensor product of the adjoint representations, so it contains only the irreducible representations with Young tableau having an integral multiple of M boxes. The same must be true for the conjugate representations for the Ψ-part. Moreover, since eq. (63) is realized on bosonic oscillators, it contains all the symmetric irreducible representations of $SU(M)$, whose Young tableau consists of a single row. Therefore, the number of boxes equals the eigenvalue of the total number operator for Ψ-oscillators. In this

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way [18], the Gauss' law plus group theory require that $MB\theta$ be an integral multiple of M, or simply

$$B\theta = \ell = 0, 1, 2, ...$$ (64)

This quantization in the Chern-Simons finite-matrix model can also be viewed from several different angles. For example, it can be understood as anomaly cancellation *a la* Callan and Harvey [21]. The anomaly here is referred as the Gauss law anomaly, i.e. $[X_1, X_2] = i\theta$, can not be satisfied by any finite matrices. Then one introduces the boundary degrees of freedom and the anomaly is cancelled or "compensated" by the boundary contributions when $B\theta$ is quantized. On the other hand, one may also consider this as the finite-matrix model version of the level quantization for the non-commutative Chern-Simons (NCCS) term. This is because the second term of the Lagrangian (49) is of the form for the NCCS density:

$$\mathcal{L} = \frac{4\pi\kappa}{3}\theta \int \epsilon_{\mu\nu\rho}\mathrm{Tr}D_\mu D_\nu D_\rho$$ (65)

with the coefficient $\kappa = B\theta/4\pi$. Here we have viewed the coordinates X_i as the covariant derivative operators: $X_i \sim \theta D_i$ and $D_0 = -i\partial_t + A_0$, a trick often used in the matrix model of D0-branes [24]. The level quantization [22, 23] requires the level $4\pi\kappa = \ell$ be integers, which is nothing but eq. (65). Finally, the same quantization has a topological origin in path integral, by requiring the complex density $\exp\{i \int dtL(t)\}$ in path integral measure to be invariant under large $U(M)$ Gauge transformations along a path with nonzero winding in the compactified temporal direction. The 1d Chern-Simons term $\mathrm{tr}A_0$ changes under large $U(1)$ transformations, so its invariance leads to the quantization (64).

The quantization condition (64) immediately indicates the quantization of the inverse filling factor, ν^{-1}. Classically $\nu^{-1} = B\theta$. However, there is a quantum correction to this identification: namely, quantum mechanically we have

$$\nu^{-1} = k = \ell + 1 = 1, 2, 3, ...\quad .$$ (66)

Certainly this is consistent with the known fact that there is a level shift, $\ell \to \ell + 1$, in quantum U(1) NCCS theory [22, 23]. For our finite-matrix model, the quantum correction is due to the operator reordering effects in the radius squared matrix, R^2, which semiclassically determines the "area" occupied by the giant gravitons. Namely upon quantization, the expression of $\mathrm{Tr}R^2$ acquires an additional "zero-point energy" term, M^2/B, according

to the quantization condition (58). This additional contribution to $\text{Tr}R^2$ increases the radius of each "semi-classical orbit" from the classical value $\sqrt{(2n-1)\theta}$, $(n = 1, 2, ..., M)$ to $\sqrt{(2n-1)(\theta + 1/B)}$. Thus, the area of the giant graviton droplet is increased due to quantum effects: Area $\simeq 2\pi M(\theta + 1/B)$ for $M \gg 1$. Since the filling factor is defined by $\nu = 2\pi M/(B \times \text{Area})$, we obtain $\nu^{-1} = \ell + 1 \equiv k$ by using the quantization condition (64).

So much for the mathematical derivation of the quantization condition (66). Here some remarks on its physical meaning are in order. The quantization condition (66) for the filling factor ν, on one hand, is similar in nature to the Dirac quantization of the monopole charge, in that the allowed values are special, directly related to the consistency of the quantum dynamics. On the other hand, it is different from the Dirac condition in that it deals with the behavior of a quantum *many-body* system as a whole. Eq. (4) together with (11) tell us that $Br_m^2/2 = \tilde{n} = $ integer, where $\tilde{n}$ is the number of the background fluxes in the area with radius r_m. Then using previous discussions one has

$$\tilde{n} = Br_M^2/2 = kM. \tag{67}$$

This result together eq. (4) yield the density of the giant gravitons on (x_1, x_2)-plane:

$$\rho_{gg} = \frac{1}{2\pi k}. \tag{68}$$

For $k = 1$, the giant graviton density is the same as that of LLM fermions in the original IQH droplet for $AdS_5 \times S^5$. However, for $k > 1$ the quantization condition (66) tells us that something special happens at the particular values (68) of the giant graviton fluid density. A monopole charge that does not satisfy the Dirac quantization condition simply can not exist. However, a system of giant gravitons with a density which does not satisfy the quantization condition (66) still exists, but they can not be in the particular states described by the ground state of the Chern-Simons matrix model. Namely self organization of giant gravitons into a new incompressible quantum fluid may happen only at the special densities given by eq. (68).

The above condition indicates that $\theta \sim l_s^2$. No further condition exists to require ν must be equal to 1, so in general the filling factor is allowed to be a fraction $\nu = 1/k$, with k an integer. Hence giant gravitons probing $AdS_5 \times S^5$ can be in a QH state with either integer or fractional filling factor, depending on the density of giant gravitons on the (x_1, x_2) plane at $y = 0$.

D. Interactions between giant gravitons

To determine the interaction between giant gravitons, we had better adopt the reduced canonical approach. At the classical level, the Gauss law constraint (51) can be solved [18] in the eigenvalue basis of X_1 by

$$\Psi = e^{-i\mu_b t}\sqrt{M\theta}|v\rangle, \qquad |v\rangle = \frac{1}{\sqrt{M}}(1, 1, ..., 1)^T,$$

$$(X_1)_{mn} = x_n\delta_{mn}, \qquad (X_2)_{mn} = y_n\delta_{mn} + \frac{i\theta}{x_m - x_n}(1 - \delta_{mn}). \tag{69}$$

This solution implies that $x_m \neq x_n$ for $m \neq n$. It is consistent with our previous argument that the world-volume gauge symmetry of the giant gravitons breaks from $U(M)$ to $U(1)^M$.

Substituting the solution (69) into the classical Hamiltonian (53), one obtains the classical Hamiltonian in terms of the variables x_m [18]:

$$H_{ca} = \frac{1}{m}\sum_{m=1}^{M}(p_m^2 + \frac{B^2}{4}x_m^2) + \frac{1}{4m}\sum_{n\neq m}\frac{(B\theta)^2}{(x_m - x_n)^2}. \tag{70}$$

H_{ca} is nothing but the integrable one-dimensional Calogero model [25] for non-relativistic particles on a line.

Then we want to quantize the system. The correct way is not to impose the canonical commutation relations directly to the variables x_n and p_n in eq. (70). We should solve the quantum version of Gauss law constraint in the eigenvalue basis of $\mathbf{X}_1$, and then substitute the solution into the Hamiltonian (53). Non-commutativity of the quantum operators shifts the strength of the two-body inverse square potential from ℓ^2 to $\ell(\ell + 1)$ [18], or in terms of the inverse filling factor k:

$$\mathbf{H}_{ca} = \frac{1}{m}\sum_{m=1}^{M}(-\frac{\partial^2}{\partial x_m^2} + \frac{B^2}{4}x_m^2) + \frac{1}{4m}\sum_{n\neq m}\frac{k(k - 1)}{(x_m - x_n)^2}. \tag{71}$$

The shift in the strength of the two-body inverse square potential is apparently related to the level shift, $\ell \to \ell + 1$, in U(1) non-commutative Chern-Simons theory [22, 23].

It is well-known that with $k = 1$ this model describes a free fermion system [26], which corresponds to the $\nu = 1$ IQH droplet in phase space and agrees with the observations made in the LLM geometry[4] and in the matrix model for AdS/CFT[6] for 1/2 BPS geometries. The essential information we obtain from the Hamiltonian (70) in the present context is that the two-body interactions between giant gravitons due to quantum fluctuations of stretched

open strings are *repulsive* for $k > 1$. In condensed matter physics, the repulsive nature of the interactions is considered to be a crucial condition for the formation of new incompressible QH fluid states. Indeed, the ground-state wave function of the Calogero model

$$\Psi_0(x_1, x_2, \cdots) = \prod_{m<n} (x_m - x_n)^k \exp\{-\frac{B}{2} \sum_n x_n^2\}, \tag{72}$$

is nothing but the 1d representation (in the Landau gauge) of the Laughlin wave function in the LLL with $\nu = 1/k$ on a disk geometry. Similar correspondence exists for excited states. (For more discussions on the explicit relationship between the Calogero model and the FQHE, see ref. [27].) Thus, from the quantum Calogero model obtained in our present context, we have demonstrated that the ground state of the QH system of giant gravitons at the right value of density, $\nu = 1/k$ with $k > 1$ integer, is in a strongly correlated quantum fluid phase, an incompressible FQH fluid of giant gravitons! (Actually k should be an odd integer; see next subsection.)

E. New Area Quantization and Stringy Exclusion Principle

To gain more insight into the FQH state of giant gravitons with $k > 1$, we note that

$$\langle n|R^2|n \rangle - \langle n-1|R^2|n-1 \rangle = 2\theta. \tag{73}$$

Thus a giant graviton in the FQH droplet with $k > 1$ occupies a k times bigger area than the $k = 1$ case. This predicts a new area quantization for the FQH state of giant gravitons. Previously in the fluid description we have already seen that the density of giant gravitons in the FQH droplet is k times bigger than that of LLM fermions in the original IQH droplet. But that is only a coarse-grained description on average. The result (73) in the matrix model is a property at the microscopic level. This microscopic property is a compelling evidence that the ground state of the Chern-Simons matrix model (49) corresponds to a new incompressible quantum fluid state, a FQH fluid state. This makes the FQH state distinct from other possible states of the system with the same fractional filling factor, say from a dilute gas phase which is compressible. We expect that the stability of the FQH state may survive from the back-reactions of the giant gravitons. Namely the new area quantization (73), which is k times bigger than that in the smooth 1/2 BPS LLM's geometries, should emerge in the new geometry that arises due to the back-reactions of giant gravitons.

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This is a new aspect of the giant graviton physics. It implies a *stringy exclusion principle* at work. This stringy exclusion principle differs from the one discussed previously [12, 28], in that it deals with a strong correlation effect of many giant gravitons, rather than a property of a single giant graviton. This stringy exclusion principle is actually an analogy of the generalized exclusion principle in condensed matter physics [29, 30] that is at work in the FQH state.

We may also examine the state (56) with quasi-hole excitations. The value of the quasi-hole charge $-q$ is arbitrary in the classical theory. It can be shown [18] that, to accompany the quantization of the filling factor ν, there is also a quantization of q, with the minimal value $q = 1/k$. By the standard treatment of the Calogero model (71), we expect that in addition to the fractional charge $q = 1/k$, the quasi-holes in the FQH liquid state of giant gravitons should have both fractional exchange (anyon) statistics with $\theta_{stat} = \pi/k$ [31, 32], and fractional exclusion statistics with $\lambda_{qh} = 1/k$. This value of exclusion statistics parameter for quasi-holes is just dual to the exclusion statistics parameter of the giant gravitons in the FQH state, $\lambda_{gg} = k$ [33, 34].

The possible values of the inverse filling factor k can be further constrained by consideration of quantum statistics. From the exchange statistics in one dimension in the sense of ref. [31] in terms of the scattering phase shift, $\exp(-i\pi k)$, k should be odd because the constituent LLM particles are fermions. The same restriction can be obtained by considering anyon statistics for quasi-hole in two dimensions, which is the same as that in the Calogero model [27]. The argument goes as follows [35]: Consider a cluster of coincident k quasi-holes. It has a charge just opposite to that of the constituent giant graviton, so it is the same as the charge of the original LLM particle. If in the spectrum the only states that carry the same charge are those of the hole of constituent giant gravitons, or the LLM particle, the cluster of coinciding quasi-holes should be identified to be an original LLM particle, including their statistical behavior. The statistics of a cluster of k anyons is known [36] to be $k^2\theta_{stat}$, which is just $k\pi$. It should be the same as that of the original LLM particle which is known to be a fermion. So we should choose

$$k = odd = 1, 3, 5, \cdots \tag{74}$$

for the giant graviton FQH fluids.

F. Back-reaction of giant gravitons: speculations

In above study of 1/2 BPS dynamics of giant gravitons in the $AdS_5 \times S^5$ background, the back-reaction of giant gravitons is neglected. This approximation is valid if the number M of the giant gravitons satisfy $M \ll \tilde{N}$; $\tilde{N}$ is the number of the R-R 5-flux in $AdS_5 \times S^5$ background. This implies that the size of the giant graviton droplet, of the order of $\sqrt{g_s M} l_s$ when all fundamental constants are restored, is much smaller than that of the IQH droplet, $\sqrt{g_s \tilde{N}} l_s$. Due to the extraordinary stability of the FQH state, one expects that the emergence of the FQH state and many of their properties, such as new area quantization and stringy exclusion principle for quasi-holes etc, may survive from the back-reactions of giant gravitons. In this subsection we speculate on this possibility.

In subsection V.A and V.C, we have seen that M giant gravitons are distributed semi-classically in a disk on the (x_1, x_2) plane with uniform density $1/(2\pi k)$. This semi-classical result is expected to survive in the full quantum theory. When $k = 1$, this QH droplet vacuum corresponds to a quasi-hole disk as a $z = 1/2$ region in the center of the $\nu = 1$ IQH droplet in the LLM geometry [4]. This recovers the known result that the back-reactions of giant gravitons in the IQH state with $k = 1$ give rise to new smooth 1/2 BPS geometries [4].

For $k > 1$ case, the droplet of giant gravitons has a k times larger area than the $k = 1$ case in the center of the LLM's IQH background on the (x_1, x_2) plane. Accordingly, the droplet of giant gravitons has a smaller density so it is a *fractional* quantum Hall fluid with giant graviton filling factor $\nu_{gg} = 1/k$. In the region of the giant graviton FQH droplet, we have actually a two-component fluid, with the second component being the original IQH background fluid, so that the total density of the original fermions in the region of the FQH droplet of the giant gravitons is $\rho = (k - 1)/(2\pi k)$. Suppose that the back-reaction of the giant gravitons in such a FQH state can be, at least in a certain decoupling limit, restricted to the 1/2 BPS sector of geometries then a new 1/2 BPS geometry in IIB supergravity will appear due to the back-reactions. Since the FQH state with $k > 1$ corresponds to a QH droplet with a region in which the density of original fermions is between 0 and 1, the corresponding LLM solutions will have a null singularity, according to refs. [4, 37, 38]. This null singularity is believed to be resolved by possible local quantum effects [39, 40, 41] and has well-defined description in dual CFT[14]. If the back-reactions in the full IIB supergravity can not be restricted semi-classically to the 1/2 BPS sector, then orbifold singularities are

possible to arise in the less-than-half BPS cases. So in general we expect that back-reactions of giant gravitons in a FQH state would lead to the emergence of geometries with (naked) singularities.

Since a FQH state is a well-behaved quantum state, which maintains long-distance quantum coherence (more precisely, algebraic off-diagonal long-range order) [42, 43], we believe quantum effects in IIB string theory on the corresponding geometry should resolve the singularity. The properties of the FQH state may help us understand the underlying mechanism in quantum gravity for the resolution of singularity.

VI. SUMMARY

In this paper, we re-examined the dynamics of giant gravitons probing $AdS_5 \times S^5$ geometry, inspired by LLM's 1/2 BPS geometry in IIB supergravity. The giant gravitons we examined are D3-branes wrapping on a three-sphere inside S^5 and, in the spirit of mini-superspace approximation for the moduli space of the 1/2 BPS geometries, their dynamics is restricted to that on a two dimensional plane in $AdS_5 \times S^5$. The non-abelian low-energy dynamics of the giant graviton probes is shown to lead to a Chern-Simons matrix model, which describes the fluid dynamics of a QH system, i.e. a system of non-relativistic charged particles moving in a strong constant magnetic field. In terms of incompressible giant graviton fluid, the gauge symmetry of the matrix model is broken from $U(M)$ down to $U(1)^M$, and an additional non-commutative constraint is supplemented to the matrix model action. The D-brane nature of the giant gravitons dictates that edge states are excited when the boundary of the giant graviton droplet fluctuates. Indeed, the low energy world-volume theory for a giant graviton fluid reduces to a Chern-Simons finite-matrix model, which is known to incorporate the edge excitations of a QH droplet of finite size, with either the integer or the fractional filling factor. We discussed various physical analogies between the giant graviton fluid and the FQH liquid, and speculated the possibility that the FQH nature of the giant graviton fluid may survive beyond the probe approximation and give rise to new geometry with singularity in IIB supergravity. In short, our results demonstrate that the giant gravitons, as quasi-hole excitations in $AdS_5 \times S^5$ droplet background, can condense into a new quantum Hall fluid with filling factor $\nu = 1/k$ for k an odd integer.

It is interesting to ask whether there is a QH analogy in the open string fluid (mentioned

in sect. IV.A), in which the $U(M)$ gauge symmetry is unbroken. The experience with the SYM matrix model[6] indicates that the answer should be "YES", because there is a one-to-one correspondence between the matrix basis (M^2 uncoupled bosonic oscillators) and the eigenvalue basis (M free fermions) in the matrix model. To show the QH analogy explicitly, we note that each of M^2 uncoupled identical oscillators behaves exactly the same as a charged particle in a magnetic field. Then we may introduce the non-commutativity constraint again by assuming the system is in an incompressible fluid state. (This assumption is self-consistent if the interactions are repulsive.) The bosonic statistics will *not* interfere with the formation of an incompressible fluid provided interactions are repulsive, which make the multi-occupied single particle states unfavorable energetically. So the system can be in an incompressible quantum fluid, analogous to bosonic Laughlin states with filling fraction $\nu = 1/k$ for even k.

One may also consider D3-brane probes wrapping on S^3 inside AdS_5. These giant gravitons are located in the exterior of the circle $r^2 = r_0^2$, and are viewed as quasi-particle excitations of the $AdS_5 \times S^5$ droplet. As charged particles, they also couple to a constant background magnetic field, but now the number of the magnetic flux is N instead of $\tilde{N}$ (see (4) and (5)). This is a system dual to the one considered in this paper. It would be interesting to study the condensation of these giant gravitons and ask whether there is a QH analogy for this system.

We conclude this paper with a short comment on the emergent QHE from the $\mathcal{N} = 4$ SYM side in AdS/CFT correspondence. Ref. [9] made a remarkable inspiring attempt in this direction. The 1/2 BPS sector for a single chiral scalar in $\mathcal{N} = 4$ SYM compactified on S^3 is a matrix model. A Chern-Simons term emerges in this matrix model if an internal circle is boosted in the bulk theory as that shown by the last equation in (9). So it is an educated guess that, following the procedures presented in this paper, we will end up with FQH systems with filling fractions of the form $1/k$ for odd k [44].

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